SPHERICAL GEOMETRY AND ITS APPLICATIONS

Qahhorova Munira Komiljon qizi
NamSU, Faculty of Physics and Mathematics 1st-cource student of Mathematics, Uzbekistan

Dilnoza Xaytmirzayevna Maxmudova
Supervisor, NamSU, Faculty of Physics and Mathematics 1st-cource student of Mathematics, Uzbekistan

ABSTRACT
Spherical geometry, a non-Euclidean branch of geometry, deals with the properties and figures on the surface of a sphere. Unlike planar geometry, it does not obey Euclid's parallel postulate. This paper investigates the foundational aspects of spherical geometry, illustrates its core theorems with mathematical formulations, and explores its real-world applications in fields such as navigation, astronomy, and geodesy. We also provide graphical representations to distinguish spherical constructions from their Euclidean counterparts.

KEYWORDS
Spherical geometry, great circles, non-euclidean geometry, navigation, astronomy, geodesy.

INTRODUCTION
Geometry, the ancient science of space and form, has evolved far beyond the flat plane of Euclid. One of the most profound extensions of classical geometry is spherical geometry, which explores figures on the surface of a sphere rather than a plane. First studied by Greek mathematicians such as Theodosius of Bithynia and later formalized in Islamic Golden Age astronomy, spherical geometry became crucial for understanding celestial motions and cartography long before the formal rise of non-Euclidean geometry in the 19th century.

In spherical geometry, the basic elements differ significantly from their Euclidean counterparts. For instance, lines are represented by great circles—the largest possible circles on a sphere's surface. The parallel postulate, a cornerstone of Euclidean geometry, fails completely: no two great circles are parallel, as they always intersect in two antipodal points. Similarly, the concept of a triangle is redefined: a spherical triangle is bounded by arcs of great circles, and its angle sum always exceeds 180°. This deviation introduces rich mathematical structures and fascinating properties.

Spherical geometry is not merely a theoretical construction—it underpins much of modern science and technology. It is the foundation for global navigation systems, satellite trajectory planning, astronomical coordinate systems, and geodesy, which models the Earth's shape for GPS and mapping technologies. In an increasingly interconnected and spatially aware world, spherical geometry enables accurate positioning, distance estimation, and directional analysis at a global scale.

In this paper, we aim to:

- Introduce the core concepts and theorems of spherical geometry,
- Compare it to Euclidean geometry through visual and algebraic examples,

- Explore its real-world applications across several scientific domains.

By examining this branch of non-Euclidean geometry through both a mathematical and practical lens, we demonstrate its central role in understanding and navigating the curved spaces of our physical world.

METHODS

To investigate spherical geometry, we adopt both analytic and geometric approaches grounded in differential geometry and classical trigonometry. Our primary model is the unit sphere in three-dimensional Euclidean space, defined by the equation:

$$x^2 + y^2 + z^2 = 1.$$

In this setting, we define the fundamental objects of spherical geometry as follows:

-**Points**: Locations on the surface of the unit sphere, each uniquely described using spherical coordinates (θ, ϕ), where:

$x = \cos\theta \sin\phi\,, y = \sin\theta \sin\phi\,, z = \cos\phi$, leq $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$.

-Lines (Geodesics): The shortest paths between two points on the sphere, which lie along great circles. Mathematically, a great circle is the intersection of the sphere with a plane that passes through the origin.

Triangles: Regions bounded by three great circle arcs. We study their properties using spherical trigonometry, particularly the Law of Sines and Law of Cosines for spherical triangles.

The angular distance between two points on the sphere, given in radians, is calculated using the dot product:

$$\cos d = u \cdot v,$$

where u *and* v are unit vectors corresponding to the two points. Since all points lie on the unit sphere, this formula directly gives the arc length of the shortest path between them.

Angles in spherical geometry are measured between the tangents to the intersecting great circles at a point. The angle sum of a spherical triangle satisfies the relation:

$$A + B + C = \pi + E,$$

where E is the spherical excess, proportional to the triangle's area.

We use modern computational tools to generate diagrams and simulate geodesics, triangle angle sums, and spherical coordinate transformations. These tools allow us to visualize how spherical geometry behaves differently from planar geometry, especially in large-scale structures.

To highlight the distinctiveness of spherical geometry, we systematically compare its theorems with those from Euclidean geometry. For example:

-**Euclidean triangle angle sum**: $A + B + C = \pi$,

-Spherical triangle angle sum: $A + B + C > \pi$.

We also explore limits: as the size of a spherical triangle becomes very small, its geometry approximates that of a Euclidean triangle-demonstrating the local flatness of spherical surfaces.

RESULTS

Our study confirms several foundational theorems of spherical geometry and highlights the striking differences from Euclidean geometry:

We constructed multiple spherical triangles on the unit sphere and verified the following:

-**Angle sum**: The angle sum $(A + B + C)$ of a spherical triangle consistently exceeded π radians (180°). For instance, a triangle formed by three great circles intersecting at right angles (e.g., between the equator and two meridians 90° apart) has three right angles:

$$A = B = C = \frac{\pi}{2} \Rightarrow A + B + C = \frac{3\pi}{2} = 270 \circ.$$

Spherical excess and area:

$$E = (A + B + C - \pi), Area = E \cdot r^2$$

For the example above, the excess is $\frac{\pi}{2}$ so the area of the triangle is $\frac{\pi}{2}$ (on a unit sphere).

We plotted shortest paths between various points using great circles and confirmed: geodesics on the sphere are arcs of great circles, between two non-antipodal points, the great circle arc represents the unique minimal path.

We tested spherical-to-Cartesian coordinate transformations and verified their consistency with the unit sphere equation. Inverses were also used to convert from Cartesian coordinates to angular ones, which is essential in mapping and astronomy. We simulated flights between cities like London and Tokyo and observed that the shortest path—modeled as a great circle—appeared as a curved arc on 2D maps, but is straight in 3D spherical space. We applied the Law of Cosines in spherical triangles to solve angular distances in navigation scenarios.

DISCUSSION

The results of our investigation demonstrate not only the theoretical elegance of spherical geometry but also its profound practical relevance. The inherent curvature of the Earth necessitates the use of spherical models in any large-scale application:

The failure of the parallel postulate in spherical geometry fundamentally reshapes the structure of space. While Euclidean lines can remain equidistant and never meet, spherical “lines” (great circles) always intersect. This property makes spherical geometry more aligned with physical reality at global scales.

Navigation: Great circle routes are standard in aviation and marine navigation, minimizing fuel and time.

Astronomy: Star charts and planetary motions are plotted on celestial spheres using spherical coordinates.

Geodesy and GPS: Earth’s geoid is modeled using spheres and ellipsoids. The accuracy of satellite positioning relies on spherical trigonometry.

Cartography: All 2D map projections involve distortions because they attempt to flatten spherical surfaces.

Spherical geometry also lays a conceptual foundation for more advanced theories:

-**Riemannian Geometry**: Spherical space serves as an example of a positively curved Riemannian manifold.

-**Relativity**: The general theory of relativity describes gravity as curvature in spacetime, and spherical models appear in Schwarzschild geometry (e.g., around black holes).

-**Education and Technology**: Incorporating spherical geometry into curricula enhances spatial reasoning and prepares students for STEM careers in physics, engineering, and computer graphics.

CONCLUSION

NEXT SCIENTISTS CONFERENCES

Spherical geometry provides a natural and essential framework for modeling curved surfaces, particularly the surface of the Earth and celestial spheres. Through mathematical exploration of its key properties-such as angle excess, the absence of parallel lines, and the behavior of geodesics-we gain both theoretical insight and practical tools.

Its applications are deeply embedded in modern technology, from satellite navigation and astronomical modeling to global communications and geographic information systems. Understanding spherical geometry equips us with the perspective needed to navigate and interpret the curved world we live in.

Future research could expand into hyperbolic and elliptic geometries, contributing to fields like cosmology and quantum gravity, where the nature of space itself is under scrutiny.

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